Definition 1.6. A point $\mathbf{x}$ is an *extreme point* of a convex set C if there exist no two distinct points $\mathbf{x}_1$ and $\mathbf{x}_2 \in C$ such that $\mathbf{x} = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ for some $\lambda \in (0,1)$.

Geometrically, extreme points are just the corner points of C.

Example 1.2. $\mathbb{R}^n$ is convex. Let W be a subspace of $\mathbb{R}^n$ and $\mathbf{x}_1, \mathbf{x}_2 \in W$. Thus any linear combination of $\mathbf{x}_1$ and $\mathbf{x}_2$ is also in W, in particular the linear combination $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in W$ for $\lambda \in [0,1]$. This shows that W is convex.

Example 1.3. The n dimensional open ball centered at $\mathbf{x}_0$ with radius r is defined as

$$B_r(\mathbf{x}_0) = \{\mathbf{x} \mid |\mathbf{x} - \mathbf{x}_0| < r\}.$$

The n dimensional closed ball centered at $\mathbf{x}_0$ with radius r is defined as

$$\overline{B_r(\mathbf{x}_0)} = \{\mathbf{x} \mid |\mathbf{x} - \mathbf{x}_0| < r\}.$$

Both the open ball and the closed ball are convex. We prove it for the open ball. Let $\mathbf{x}_1, \mathbf{x}_2 \in B(\mathbf{x}_0)$ and $\lambda \in [0,1]$. Then

$$\begin{aligned} |(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2) - \mathbf{x}_0| &= |\lambda(\mathbf{x}_1 - \mathbf{x}_0) + (1-\lambda)(\mathbf{x}_2 - \mathbf{x}_0)| \\ &\le \lambda|\mathbf{x}_1 - \mathbf{x}_0| + (1-\lambda)|\mathbf{x}_2 - \mathbf{x}_0| \\ &\le \lambda r + (1-\lambda) r = r. \end{aligned}$$

iff

Let S be a subset of $\mathbb{R}^n$. A point $\mathbf{x}$ is a *boundary point* of S if every open ball centered at $\mathbf{x}$ contains both a point in S and a point in $\mathbb{R}^n - S$. Note that a boundary point can either be in S or not in S. The set of all boundary points of S, denoted by ∂S, is the *boundary* of S. A set S is *closed* if $\partial S \subset S$. A set S is open if its complement $\mathbb{R}^n - S$ is closed. Note that a set that is not closed is *not* necessarily open; and a set that is not open is *not necessarily closed.* There are sets that are neither open nor closed. The *closure* of a set S is the set $\overline{S} = S \cup \partial S$. The *interior* of a set S is the set $S^\circ = S - \partial S$. A set S is closed if and only if $S = \overline{S}$. A set S is open if and only if $S = S^\circ$.

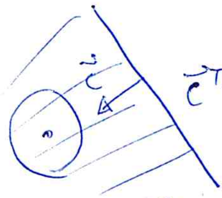
Example 1.4. $\mathbb{R}^n$ is both open and closed. The empty set $\emptyset$ is both open and closed.

Example 1.5. The hyperplane $P = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = z\}$ is closed in $\mathbb{R}^n$. In fact we will show that $P \subseteq \partial P$. Without loss of generality we may assume $|\mathbf{c}| = 1$. Let $\mathbf{x} \in P$ and $B_r(\mathbf{x})$ is an open ball centered at $\mathbf{x}$ with radius r. Since $\mathbf{x} \in B_r(\mathbf{x})$ it remains to show that $B_r(\mathbf{x})$ contains a point not in P. Let

$$\mathbf{y} = \mathbf{x} + \frac{r}{2}\mathbf{c}$$

then

$$\mathbf{c}^T\mathbf{y} = \mathbf{c}^T\mathbf{x} + \frac{r}{2}\mathbf{c}^t\mathbf{c} = z + \frac{r}{2} > z.$$



$\vec{c}^T\vec{x} = z$

Hence $\mathbf{y} \notin P$. But $|\mathbf{y} - \mathbf{x}| = \frac{r}{2}$ therefore $\mathbf{y} \in B_r(\mathbf{x}$.

Example 1.6. The half spaces

$\partial P \subseteq P$

$$X_1 = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} \le z\} \quad \text{and} \quad X_2 = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} \ge z\}$$

$\vec{c}^T\vec{x} \geq z$

are closed in $\mathbb{R}^n$. In fact we have $\partial X_1 = \partial X_2 =$ the hyperplane $P = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = z\}$. We will show that $\partial X_1 = P$, the proof for $\partial X_2 = P$ is similar.

($P \subset \partial X_1$) Let $\mathbf{x} \in P$. For any $r > 0$, let

$$\mathbf{y}_1 = \mathbf{x} + \frac{r}{2|\mathbf{c}|}\mathbf{c}, \quad \mathbf{y}_2 = \mathbf{x} - \frac{r}{2|\mathbf{c}|}\mathbf{c}.$$

We see that $|\mathbf{x} - \mathbf{y}_1| = \frac{r}{2} = |\mathbf{x} - \mathbf{y}_2|$ so both $\mathbf{y}_1, \mathbf{y}_2 \in B_r(\mathbf{x})$. Moreover

$$\mathbf{c}^T\mathbf{y}_1 = \mathbf{c}^T\mathbf{x} + \frac{r}{2|\mathbf{c}|}\mathbf{c}^T\mathbf{c} = r + \frac{r}{2} > r$$

and therefore $\mathbf{y}_1 \notin X_1$. On the other hand

$$\mathbf{c}^T\mathbf{y}_2 = \mathbf{c}^T\mathbf{x} - \frac{r}{2|\mathbf{c}|}\mathbf{c}^T\mathbf{c} = r - \frac{r}{2} < r$$

so $\mathbf{y}_2 \in X_1$. This shows $\mathbf{x} \in \partial X_1$.

($\partial X_1 \subset P$) Suppose $\mathbf{x} \notin P$. If $\mathbf{c}^T\mathbf{x} = z_1 < z$ then let $r = \frac{z-z_1}{2} > 0$. The open ball $B_r(\mathbf{x})$ lies entirely in X_1. So $B_r(\mathbf{x})$ contains no point outside of X_1, hence $\mathbf{x} \notin \partial X_1$. If $\mathbf{c}^T\mathbf{x} = z_1 > z$ then let $r = \frac{z_1-z}{2} > 0$. The open ball $B_r(\mathbf{x})$ lies entirely outside of X_1. So $B_r(\mathbf{x})$ contains no point of X_1, hence $\mathbf{x} \notin \partial X_1$. In either case $\mathbf{c} \notin \partial X_1$.

$\{\vec{c}^T\vec{x} = z\}$ $\quad$ $\vec{c}^T\vec{x}_1 \le z$ $\quad$ $\vec{c}^T\vec{x}_2 \le z$

Lemma 1.1. *(a) All hyperplanes are convex.*

$c^T(\lambda\vec{x}_1 + (1-\lambda)\vec{x}_2) \le z$

(b) The closed half-space $\{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} \le z\}$ and $\{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} \ge z\}$ are convex.

$\lambda \in [0,1]$

(c) The open half-space $\{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} < z\}$ and $\{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} > z\}$ are convex.

(d) Any intersection of convex sets is still convex.

(e) The set of all feasible solutions *to a linear programming problem is a convex set.*

Proof. (a) Let $X = \{\mathbf{x} \mid \mathbf{c}^T\mathbf{x} = z\}$ be our hyperplane. For all $\mathbf{x}_1, \mathbf{x}_2 \in X$ and $\lambda \in [0,1]$, we have

$$\mathbf{c}^T[\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2] = \lambda\mathbf{c}^T\mathbf{x}_1 + (1-\lambda)\mathbf{c}^T\mathbf{x}_2 = \lambda z + (1-\lambda)z = z.$$

Thus $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in X$. Hence X is convex.

(b) and (c) can be proved similarly by replacing the equality signs in (a) by the corresponding inequality signs.

(d) Let $C = \bigcap_{\alpha \in I} C_\alpha$, where C_α are convex for all α in the index set I. Then for all $\mathbf{x}_1, \mathbf{x}_2 \in C$, we have $\mathbf{x}_1, \mathbf{x}_2 \in C_\alpha$ for all $\alpha \in I$. Hence for all $\lambda \in [0,1]$,

$$\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C_\alpha$$

for all $\alpha \in I$. Thus $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in C$, and C is convex.

(e) For any LP problem, the constraints can be written as $\mathbf{a}_i\mathbf{x} \le b_i$ or $\mathbf{a}_i\mathbf{x} = b_i$ etc. The set of points that satisfy any one of these constraints is thus a half space or a hyperplane. By (a), (b) and (c), they are convex. By (d), the intersection of all these sets, which is defined to be the set of feasible solutions, is a convex set. $\square$

Definition 1.7. Let $\{x_1, \cdots x_k\}$ be a set of given points. Let

$$\mathbf{x} = \sum_{i=1}^{k} M_i\mathbf{x}_i$$

where $M_i \ge 0$ for all i and $\sum_{i=1}^{k} M_i = 1$. Then $\mathbf{x}$ is called a *convex combination* of the points $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k$.

Example 1.7. Consider the triangle on the plane with vertices $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$. Then any point $\mathbf{x}$ in the triangle is a convex combinations of $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$. In fact, let $\mathbf{y}$ be the extension of the line segment

Closure $S \cup \partial S = \bar{S}$

interior $S \setminus \partial S = S^\circ$

Facts: (i) S is closed iff $S = \bar{S}$
(ii) S is open iff $S = S^\circ$
(iii) $\bar{S}$ must be closed
(iv) S° must be open.

pf: (i) ($\Rightarrow$) Suppose S is closed

(a) $S \subseteq \bar{S}$
(b) S is closed, $\therefore$ $\partial S \subseteq S \subseteq \bar{S}$

$$\bar{S} \overset{(def)}{=} S \cup \partial S \overset{(a)(b)}{\subseteq} S \overset{(a)}{\subseteq} \bar{S}$$

$$\Rightarrow \bar{S} = S.$$

(ii) ($\Leftarrow$) if $S = \bar{S}$ show that S is closed ($\partial S \subseteq S$)

$$\partial S \subseteq \bar{S} = S \Rightarrow \partial S \subseteq S. \#$$

(iii) $\partial \bar{S} \subseteq \bar{S}$.

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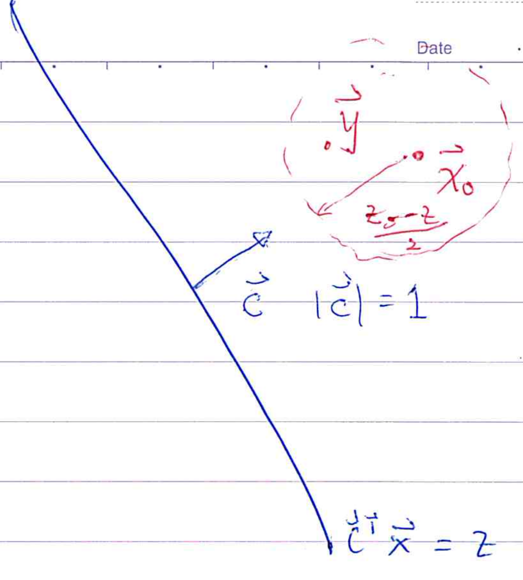
$$\vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \quad |\vec{c}| = \sqrt{c_1^2 + \cdots + c_n^2}$$



Eg 1.5 $\quad P = \{ x \mid \vec{c}^T \vec{x} = z \}$ is closed

" $\partial P \subseteq P$ "

$\Leftrightarrow$ " $\forall \vec{x} \in \partial P$ then $\vec{x} \in P$ "

$\Leftrightarrow$ " $\forall \vec{x} \notin P$ then $\vec{x} \notin \partial P$ "

pf: $\quad x_0 \notin P \Rightarrow \vec{c}^T \vec{x}_0 \neq z$

w.l.o.g. $\quad \vec{c}^T \vec{x}_0 = z_0 > z$

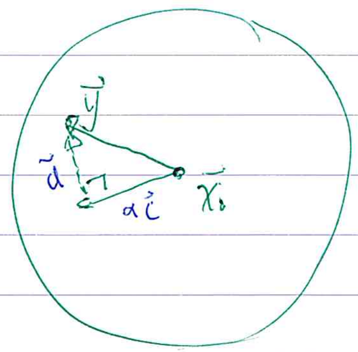
without loss of generality

geometrically " $B_{\left(\frac{z_0 - z}{2}\right)}(\vec{x}_0) \subseteq P^c$ "

$\forall \vec{y} \in B_{\frac{z_0 - z}{2}}(\vec{x}_0)$ want to show that $\vec{y} \notin P$

(i) $\quad |\vec{y} - \vec{x}_0| < \frac{z_0 - z}{2}$

(ii) $\quad \vec{y} - \vec{x}_0 = \alpha \vec{c} + \vec{d}$ s.t. $\vec{d} \perp \vec{c}$



(iii) ($\Delta \geq$) $\quad |\vec{y} - \vec{x}_0| \geq |\alpha \vec{c}| = |\alpha|$

(iv) $\quad \vec{c}^T(\vec{y} - \vec{x}_0) = \vec{c}^T(\alpha \vec{c} + \vec{d}) = \alpha |\vec{c}|^2 = \alpha$

(v) $\quad \vec{c}^T \vec{y} = \vec{c}^T \vec{x}_0 + \alpha = z_0 + \alpha > z_0 - \frac{z_0 - z}{2}$

$\quad = \frac{z_0 + z}{2} > z$ # $\quad (\because z_0 > z)$

$\vec{c}^T \vec{y} > z \Rightarrow \vec{y} \notin P$

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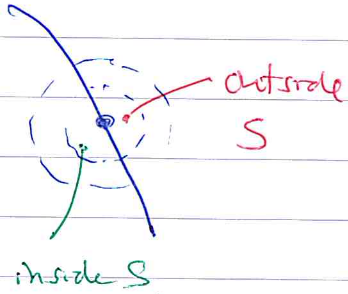
n $\forall$ for all = $\exists$ there exist.

Boldary pt if $\vec{x} \in \partial S$



then $\forall B_r(\vec{x})$ (i.e. $\forall r \in \mathbb{R}^+$)

(i) $\exists \vec{x}_1 \in B_r(\vec{x})$ & $\vec{x}_1 \in S$

and (ii) $\exists \vec{x}_2 \in B_r(\vec{x})$ & $\vec{x}_2 \notin S$

Not a boldary pt if $\vec{x} \notin \partial S$

then $\exists B_r(\vec{x})$ $r>0$

(i) $\forall \vec{x}_1 \in B_r(\vec{x})$ $\vec{x}_1 \in S$ $(\Leftrightarrow B_r(\vec{x}) \subseteq S)$

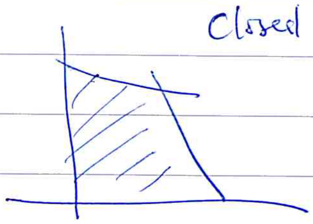
or (ii) $\forall x_1 \in B_r(\vec{x})$ $\vec{x}_2 \notin S$ $(\Leftrightarrow B_r(\vec{x}) \subseteq S^c)$

Closed $\partial S \subseteq S$

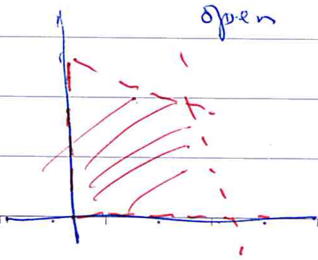
open ~~$\partial \subseteq S$~~ $\partial(S^c) \subseteq S^c$

$$\begin{cases} x_1 + x_2 \le 50 \\ 40x_1 + 60x_2 \le 2400 \\ x_1 \ge 0,\ x_2 \ge 0 \end{cases} \qquad \begin{cases} x_1 + x_2 < 50 \\ 40x_1 + 60x_2 < 2400 \\ x_1 > 0 \quad x_2 > 0 \end{cases} \qquad \begin{cases} x_1 + x_2 \le 50 \\ 40x_1 + 60x_2 \le 2400 \\ x_1 \ge 0 \quad x_2 > 0 \end{cases}$$

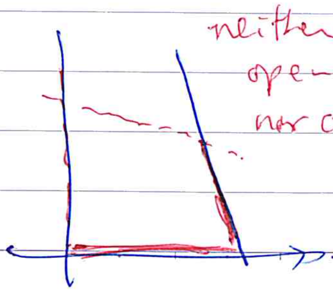
Closed



open



neither open nor close



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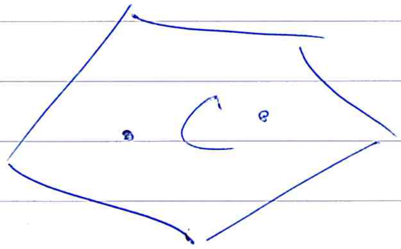

$\forall \vec{x}_1, \vec{x}_2 \in C$

then $\lambda \vec{x}_1 + (1-\lambda)\vec{x}_2 \in C$

$\lambda \in [0,1]$

(d) $\bigcap_\alpha C_\alpha$ is convex if $\{C_\alpha\}$ are convex

pf: $\vec{x}_1 \in \bigcap_\alpha C_\alpha \quad \vec{x}_2 \in \bigcap_\alpha C_\alpha \overset{?}{\Rightarrow} \lambda \vec{x}_1 + (1-\lambda)\vec{x}_2 \in \bigcap_\alpha C_\alpha$

$\Downarrow \qquad \Downarrow$

$\vec{x}_1 \in C_\alpha, \forall \alpha \qquad x_2 \in C_\alpha \ \forall \alpha$

$\Downarrow$

$\lambda x_1 + (1-\lambda) x_2 \in C_\alpha \ \forall \alpha$ ($\because$ every C_α ~~are~~ is convex)

$\lambda \in [0,1]$

$\Downarrow$

$\lambda x_1 + (1-\lambda) x_2 \in \bigcap_\alpha C_\alpha$

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